

# FRACTAL ENTROPIES AND DIMENSIONS FOR MICROSTATE SPACES, II

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**ABSTRACT.** For a selfadjoint element  $x$  in a tracial von Neumann algebra and  $\alpha = \delta_0(x)$  we compute bounds for  $\mathbb{H}^\alpha(x)$ , where  $\mathbb{H}^\alpha(x)$  is the free Hausdorff  $\alpha$ -entropy of  $x$ . The bounds are in terms of  $\int \int_{\mathbb{R}^2 - D} \log |y - z| d\mu(y) d\mu(z)$  where  $\mu$  is the Borel measure on the spectrum of  $x$  induced by the trace and  $D \subset \mathbb{R}^2$  is the diagonal. We compute similar bounds for the free Hausdorff entropy of a free family of selfadjoints.

## INTRODUCTION

[1] introduced fractal geometric entropies and dimensions for Voiculescu's microstate spaces ([3], [4]). One can associate to a finite set of selfadjoint elements  $X$  in a tracial von Neumann algebra and an  $\alpha > 0$  an extended real number  $\mathbb{H}^\alpha(X) \in [-\infty, \infty]$ .  $\mathbb{H}^\alpha(X)$  is a kind of asymptotic logarithmic  $\alpha$ -Hausdorff measure of the microstate spaces of  $X$ . One can also define a free Hausdorff dimension of  $X$ , denoted by  $\mathbb{H}(X)$ , which is related to  $\mathbb{H}^\alpha(X)$  in the same way that Hausdorff dimension is related to the critical value of Hausdorff measures.  $\mathbb{H}^\alpha$  can be regarded as an interpolated version of Voiculescu's free entropy  $\chi$  in the sense that if  $X$  consists of  $n$  selfadjoints, then  $\mathbb{H}^n(X) = \chi(X) + \frac{n}{2} \log(\frac{2n}{\pi e})$ . This connection seems perfectly natural since  $\chi$  is defined in terms of Lebesgue measure and Hausdorff  $n$  measure is a normalization of  $n$ -dimensional Lebesgue measure.

In [3] Voiculescu establishes an equation for  $\chi(x)$  where  $x$  is a selfadjoint operator. He shows that if  $\mu$  is the Borel measure on  $sp(x)$  induced by the tracial state, then the free entropy of  $x$  is a normalization of the logarithmic energy of  $\mu$ , i.e.,

$$\chi(x) = \int \int \log |y - z| d\mu(y) d\mu(z) + \frac{3}{4} + \frac{1}{2} \log 2\pi.$$

Moreover, Voiculescu showed in the same work that if  $\{x_1, \dots, x_n\}$  is a free family of selfadjoints, then  $\chi(x_1, \dots, x_n) = \chi(x_1) + \dots + \chi(x_n)$ .

It is natural to wonder whether similar properties hold for the free Hausdorff  $\alpha$ -entropy. The strongest statement in this direction might go as follows. If  $x$  is selfadjoint and  $\alpha$  is the free Hausdorff dimension of  $x$ , then

$$\mathbb{H}^\alpha(x) = \int \int_{\mathbb{R}^2 - D} \log |y - z| d\mu(y) d\mu(z) + K_\alpha,$$

where  $K_\alpha$  is some constant dependent on  $\alpha$  and  $D$  is the diagonal line in  $\mathbb{R}^2$ . Using Voiculescu's strengthened asymptotic freeness results, free additivity would follow. If this is too much to ask for, then one might hope to show that  $\mathbb{H}^\alpha(x)$  is bounded in terms of  $\int \int_{\mathbb{R}^2 - D} \log |y - z| d\mu(y) d\mu(z)$ . Presumably, this would be followed by showing that a free family of selfadjoints has Hausdorff entropy proportional to the sums of the free Hausdorff entropies of each element. Unfortunately, due to technical difficulties and some fundamental differences between  $\mathbb{H}^\alpha$  and  $\chi$ , neither equations nor estimates of these kinds were present in [1].

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The present work is an addendum to [1] where we fill in this gap by showing the weaker of the two proposed problems, namely for  $\alpha = \mathbb{H}(x)$  there exist constants  $K_1, K_2$  dependent only on  $\alpha$  such that

$$K_1 \leq \int \int_{\mathbb{R}^2 - D} \log |y - z| d\mu(y) d\mu(z) - \mathbb{H}^\alpha(x) \leq K_2.$$

Moreover, we show that these bounds promote to ones for a freely independent, finite family of selfadjoints.

The techniques are very much in the spirit of those in part 1. Because all the microstate spaces are naturally associated to locally isometric spaces all the Hausdorff quantities can be bound with strong packing estimates and these in turn can be computed by results of Mehta. The only new aspects involve sharpening the aforementioned methods to arrive at tighter estimates.

There are five short sections. The first is a list of notation. The second is a brief collection of properties we will use about microstates for a single selfadjoint. The third and fourth sections are the upper and lower bounds, respectively, on the free Hausdorff entropy of a single selfadjoint, and the fifth section deals with the free situation.

## 1. NOTATION

Throughout suppose  $M$  is a von Neumann algebra with faithful, tracial state  $\varphi$ . Suppose  $x = x^* \in M$  and  $\mu$  is the Borel measure on  $sp(x)$  induced by  $\varphi$ .  $\alpha = \mathbb{H}(x)$ , which by [1] and [4] is just  $\delta_0(x) = \delta(x)$ . Suppose  $R > \|x\|$ . We maintain the standard notation introduced in [4] for the microstate spaces.  $M_k^{sa}(\mathbb{C})$  denotes the set of selfadjoint  $k \times k$  complex matrices and  $(M_k^{sa}(\mathbb{C}))^n$  is the space of  $n$ -tuples with entries in  $M_k^{sa}(\mathbb{C})$ . All metric quantities for the microstate spaces will be taken with respect to the norm  $|\cdot|_2$  on  $(M_k^{sa}(\mathbb{C}))^n$  given by  $|(a_1, \dots, a_n)|_2 = (\sum_{j=1}^n tr_k(a_j^2))^{\frac{1}{2}}$  where  $tr_k$  is the tracial state on  $M_k(\mathbb{C})$ .  $vol$  denotes Lebesgue measure on  $(M_k^{sa}(\mathbb{C}))^n$  with respect to the Hilbert space norm  $\sqrt{k} \cdot |\cdot|_2$  and  $L_k$  denotes the Lebesgue measure of the ball of radius  $\sqrt{k}$  in  $\mathbb{R}^{k^2}$ .  $D$  will denote the diagonal line in  $\mathbb{R}^2$ .

## 2. MICROSTATES FOR A SINGLE SELFADJOINT

Because the von Neumann algebra generated by  $x$  is hyperfinite, the microstate space for a single selfadjoint  $x$  is obtained by taking unitary orbit of one well-approximating microstate. Since the estimates involve the entropy (and not dimension), we will need a sharper handle on such ‘well-approximating’ microstates and define two kinds of microstates for  $x$ :  $A_k$  and  $B_k$ . The  $A_k$  will be used for the upper bound and the  $B_k$  will be used for the lower bound.

Write  $\mu = \sigma + \nu$  where  $\sigma$  is the atomic part of  $\mu$  and  $\nu$  is the diffuse part of  $\mu$ .  $\sigma = \sum_{i=1}^s c_i \delta_{r_i}$  for some  $s \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ ,  $c_i \geq c_{i+1} > 0$ , and where for  $i \neq j$ ,  $r_i \neq r_j$ . Set  $c = \nu([a, b])$  where  $sp(x) \subset [a, b]$ . Because  $\nu$  is diffuse for each  $k$  and  $1 \leq j \leq [ck]$  there exists a largest number  $\lambda_{jk} \in [a, b]$  satisfying  $\nu([a, \lambda_{jk}]) = \frac{j}{k}$ .

**2.1.  $A_k$  microstates for the upper bound.** For each  $k$  denote by  $A_k$  the  $k \times k$  diagonal matrix obtained by filling in the first  $[ck]$  entries with  $\lambda_{1k}, \dots, \lambda_{[ck]k}$  and the last  $\sum_{j=1}^s [c_j k]$  diagonal entries filled with  $r_1$  repeated  $[c_1 k]$  times,  $r_2$  repeated  $[c_2 k]$  times, etc., in that order. Fill in the remaining  $k - [ck] - \sum_{i=1}^s [c_i k]$  terms with 0's. Observe that for any  $m \in \mathbb{N}$  and  $\gamma > 0$   $A_k \in \Gamma_R(x; m, k, \gamma)$  for  $k$  sufficiently large. Also observe that if  $\epsilon > 0$ , and  $a_{1k}, \dots, a_{kk}$  are the eigenvalues of  $A_k$  ordered from least to greatest and according to multiplicity, then

$$\lim_{k \rightarrow \infty} k^{-2} \log \prod_{1 \leq i < j \leq k} ((a_{ik} - a_{jk})^2 + \epsilon) = \int \int \log(|y - z|^2 + \epsilon) d\mu(y) d\mu(z).$$

This follows by writing each term in the limit as the integral of an obvious simple function  $f_k(y, z)$  defined on  $[a, b]^2$ .  $\langle f_k \rangle_{k=1}^\infty$  will be a sequence uniformly bounded by  $\max\{|\log \epsilon|, |\log(b-a)|\}$  and  $f_k(s, t) \rightarrow \log((y-z)^2 + \epsilon)$  a.e.  $\mu \times \mu$ . It then follows from Lebesgue's Dominated Convergence Theorem that

$$k^{-2} \log \Pi_{1 \leq i < j \leq k} ((a_{ik} - a_{jk})^2 + \epsilon) = \int \int f_k(y, z) d\mu(y) d\mu(z) \rightarrow \int \int \log(|y-z|^2 + \epsilon) d\mu(y) d\mu(z).$$

**2.2.  $B_k$  microstates for the lower bound.** The  $B_k$  are defined only when  $s \geq 1$ , i.e.,  $\mu$  has a nontrivial atomic part, and when  $\int \int_{\mathbb{R}^2-D} \log |y-z| d\mu(y) d\mu(z)$  is finite. In this case  $B_k$  will be the  $k \times k$  diagonal matrix obtained by first adding  $r_1 [c_1 k] - \sqrt{k}$  times, followed by adding  $r_2 [c_2 k]$  times, then  $r_3$  added  $[c_3 k]$  times, and continuing in this way. This process will terminate for there exists a maximum value  $N_k \in \mathbb{N}$  dependent on  $k$  for which if  $j > N_k$  then  $[c_j k] = 0$ . Now recall the  $\lambda_{jk}$  defined in the preceding paragraph. For each  $1 \leq m \leq N_k$  find the largest  $\lambda_{jk}$  less than or equal to  $r_j$  and the smallest  $\lambda_{jk}$  greater than or equal to  $r_j$ . Denote by  $R_k$  the set of all such  $\lambda_{jk}$ . Observe that  $\#R_k \leq 2N_k$ . Fill in the remaining entries of  $B_k$  with  $\{\lambda_{2k}, \dots, \lambda_{([ck]-1)k}\} - R_k$ , ordered from greatest to least. This leaves a remaining  $F_k$  entries to fill in where  $F_k \leq 2N_k + \sum_{j=N_k+1}^k c_j k$ . Fill these entries with  $B + 3 + \frac{1}{F_k}, B + 3 + \frac{2}{F_k}, \dots, B + 4$ . For any given  $m \in \mathbb{N}$  and  $\gamma > 0$ ,  $B_k \in \Gamma_{R+4}(x; m, k, \gamma)$  for  $k$  sufficiently large. Let  $b_{1k}, \dots, b_{kk}$  be the eigenvalues of  $B_k$  ordered from least to greatest and with respect to multiplicity.

Denote by  $S_k$  the set of all 2-tuples  $(i, j)$  such that  $1 \leq i < j \leq k$ , and  $b_{ik} = b_{jk}$ ; this can only happen when  $b_{ik} = b_{jk}$  is one of the atoms  $r_1, \dots, r_{N_k}$  of  $\sigma$ . Denote by  $W_k$  the set of all 2-tuples  $(i, j)$  such that  $1 \leq i < j \leq k$ , and  $(i, j) \notin S_k$ . I claim that

$$(1) \quad \liminf_{k \rightarrow \infty} k^{-2} \cdot \sum_{(i,j) \in W_k} \log(b_{ik} - b_{jk})^2 \geq \int \int_{\mathbb{R}^2-D} \log |y-z| d\mu(y) d\mu(z).$$

Fix  $k$ . Define  $X_k$  to consist of all  $(i, j) \in W_k$  such that  $1 \leq i < j \leq k$ , either  $b_{ik}$  or  $b_{jk}$  does not belong to  $\{r_1, \dots, r_{N_k}\}$ , and satisfying the condition that  $a \leq b_{ik}, b_{jk} \leq b$ . If  $b_{ik}$  is not an element of  $\{\lambda_{1k}, \dots, \lambda_{[ck]k}\}$ , then denote by  $R(i)$  the smallest element in  $\{\lambda_{1k}, \dots, \lambda_{[ck]k}\}$  larger than  $b_{ik}$  and  $L(i)$  the largest element in  $\{\lambda_{1k}, \dots, \lambda_{[ck]k}\}$  smaller than  $b_{ik}$ . If neither  $a_{ik}$  nor  $b_{jk}$  belong to  $\{r_1, \dots, r_{N_k}\}$  then set  $f_{ij}$  to be  $\log |b_{ik} - b_{jk}|^2$  times the characteristic function over the set  $((b_{ik}, R(i)) - \{r_j : 1 \leq j \leq s\}) \times ((L(j), b_{jk}) - \{r_j : 1 \leq j \leq s\})$ . Otherwise either  $b_{ik} \in \{r_1, \dots, r_{N_k}\}$  or  $b_{jk} \in \{r_1, \dots, r_{N_k}\}$  but not both. In the former case define  $f_{ij}$  to be  $\frac{1}{c_p k} \cdot \log |a_{jk} - r_p|^2$  times the characteristic function over  $\{r_p\} \times ((L(j), b_{jk}) - \{r_j : 1 \leq j \leq s\})$  where  $r_p = b_{ik}$ . Observe that  $r_p < L(j)$ . In the latter case define  $f_{ij}$  to be  $\frac{1}{c_p k} \cdot \log |b_{ik} - r_p|^2$  times the characteristic function over  $((b_{ik}, R(i)) - \{r_j : 1 \leq j \leq s\}) \times \{r_p\}$  where  $r_p = b_{jk}$ . Observe that  $R(i) < r_p$ . Finally, set

$$g_k(y, z) = \sum_{1 \leq i < j \leq N_k} \log |r_i - r_j| \cdot \chi_{(\{r_i\} \times \{r_j\})}(y, z) + \sum_{(i,j) \in X_k} f_{ij}(y, z).$$

Because for each  $(i, j) \in X_k$ ,  $\int \int f_{ij}(y, z) d\mu(y) d\mu(z) = k^{-2} \cdot \log(b_{ik} - b_{jk})^2$  it is clear that

$$(2) \quad \int \int_{r < t} g_k(y, z) d\mu(y) d\mu(z) \leq \sum_{1 \leq i < j \leq k} k^{-2} \log(b_{ik} - b_{jk})^2 - \sum_{1 \leq i < j \leq N_k} k^{-2} \log \left( \frac{j-i}{k} \right).$$

Notice that the second term in the sum on the right hand side above converges to 0 as  $k \rightarrow \infty$  since  $\lim_{k \rightarrow \infty} \frac{N_k}{k} = 0$ .  $|g_k(y, z)| \leq \max\{|\log(y-z)^2|, |\log(b-a)|\}$  for any  $k$ .

Moreover,  $g_k(y, z) \rightarrow \log(y - z)^2$  a.e.  $\mu \times \mu$  on the region  $E$  strictly above  $D$ . To see this suppose  $\epsilon, L > 0$  and choose  $n \in \mathbb{N}$  so large that  $\sum_{j=n}^s c_j < \epsilon$ . Write  $K = \text{support}(\nu) - \{r_j : 1 \leq j \leq s\}$ . There exists a  $\delta > 0$  such that the  $\mu \times \mu$  measure of  $\{(y, z) \in \mathbb{R}^2 : 0 < |y - z| < \delta\}$  is no greater than  $\epsilon$ . Define  $T$  to be the intersection of the region strictly above the line  $y = z + \delta$  and

$$(\{r_j : 1 \leq j \leq n\} \cup K)^2.$$

$(\nu \times \nu)(E - T) < \epsilon$ . Define for each  $k$ ,  $G_k = \{1 \leq i \leq [ck] : |\lambda_{ik} - \lambda_{jk}| < \frac{\delta}{L}, j = i \pm 1\}$ . It is each to show that  $\lim_{k \rightarrow \infty} \#G_k / ck = 1$ . If  $H_k$  consists of all  $b_{jk}$  such that  $b_{jk} = \lambda_{ik}$  for some  $i \in G_k$ , then  $\lim_{k \rightarrow \infty} \#H_k / ck = 1$ . Write  $I_k$  for the union of all Cartesian products of the form  $((b_{ik}, R(i)) - \{r_j : 1 \leq j \leq s\}) \times ((L(j), b_{jk}) - \{r_j : 1 \leq j \leq s\})$ ,  $\{r_p\} \times ((L(j), b_{jk}) - \{r_j : 1 \leq j \leq s\})$  (when  $r_p = b_{ik}$ ), and  $((b_{ik}, R(i)) - \{r_j : 1 \leq j \leq s\}) \times \{r_p\}$  (when  $r_p = b_{ik}$ ) where  $(i, j) \in X_k \cap H_k^2$ . Take all the union of all these sets with  $\{r_j : 1 \leq j \leq n\}^2$  and call the resultant set  $I_k$ . Observe that  $\lim_{k \rightarrow \infty} (\mu \times \mu)(T - I_k) = 0$  and that on  $I_k$   $g_k(y, z)$  and  $\log(y - z)^2$  differ by no more than  $\frac{\delta}{L} \cdot \max\{|\log \delta|, |\log(b - a)|\}$ . As  $L$  was arbitrary, it follows that  $g_k(y, z)$  converges to  $\log(y - z)^2$  almost everywhere  $\mu \times \mu$  on  $T$ . Since  $\epsilon > 0$  was arbitrary and  $(\mu \times \mu)(E - T) < \epsilon$ ,  $g_k(y, z)$  converges to  $\log(y - z)^2$  on  $E$  a.e.  $\mu \times \mu$ .

It now follows from Lebesgue's Dominated Convergence Theorem that

$$\int \int_E g_k(y, z) d\mu(y) d\mu(z) \rightarrow \int \int_E \log(y - z)^2 d\mu(y) d\mu(z) = \int \int_{\mathbb{R}^2 - D} \log |y - z| d\mu(y) d\mu(z).$$

Combining this with the preceding inequality gives (1).

### 3. UPPER BOUND

**Lemma 3.1.**  $\mathbb{H}^\alpha(x) \leq \int \int_{\mathbb{R}^2 - D} \log |s - t| d\mu(s) d\mu(t) + \log 16 + \frac{1}{4}$ .

*Proof.* Suppose  $\epsilon, t > 0$  are given. There exist  $m \in \mathbb{N}$  and  $\gamma > 0$  such that for any  $k \in \mathbb{N}$  and  $A, B \in \Gamma(x; m, k, \gamma)$  there exists a unitary  $u$  satisfying  $|uAu^* - B|_2 < t$ . Consider the sequence  $\langle A_k \rangle_{k=1}^\infty$  constructed in Lemma 2.1. For sufficiently large  $k$ ,  $A_k \in \Gamma(x; m, k, \gamma)$  and moreover,

$$\begin{aligned} H_\epsilon^{\alpha k^2}(\Gamma(x; m, k, \gamma)) &\leq H_\epsilon^{\alpha k^2}(\Theta_t(A_k)) \leq K_{\frac{\epsilon}{2}}(\Theta_t(A_k)) \cdot \epsilon^{\alpha k^2} \leq P_{\frac{\epsilon}{4}}(\Theta_t(A_k)) \cdot \epsilon^{\alpha k^2} \\ &\leq \text{vol}(\Theta_{t+\epsilon/4}(A_k)) \cdot \frac{4^{k^2} \epsilon^{(\alpha-1)k^2}}{L_k}. \end{aligned}$$

By Lemma 4.2 of [3]  $\text{vol}(\Theta_{t+\epsilon/4}(A_k))$  is dominated by

$$k^{k/2} \cdot \epsilon^k \cdot \Gamma\left(\frac{k}{2} + 1\right)^{-1} \cdot (1 + 2\alpha)^{k(k-1)/2} \cdot e^{2k^2\epsilon} \cdot \pi^{k^2/2} \cdot 2^{k(k-1)/2} \cdot (\Pi_{j=1}^k j!)^{-1} \cdot \Pi_{1 \leq i < j \leq k} ((a_{ik} - a_{jk})^2 + \epsilon).$$

Here  $\alpha \in (0, \frac{1}{2})$  is the unique number such that  $(\alpha + 2\alpha^2)(\alpha + 2)^{-1})^{1/2} = \frac{t}{\epsilon} + \frac{1}{4}$ . Thus, using Lemma 4.4 of [3] and the preliminary remarks in Section 2 it follows that

$$\begin{aligned}
 \mathbb{H}_\epsilon^\alpha(x) &\leq \mathbb{H}_\epsilon^\alpha(x; m, \gamma) \\
 &\leq \limsup_{k \rightarrow \infty} k^{-2} \cdot [\log(\text{vol}(\Theta_{t+\epsilon/4}(A_k)) - \log L_k) + \log 4 + (\alpha - 1) \log \epsilon] \\
 &\leq \limsup_{k \rightarrow \infty} k^{-2} \cdot \log \Pi_{1 \leq i < j \leq k} ((a_{ik} - a_{jk})^2 + \epsilon) + \log 16 + (\alpha - 1) \log \epsilon + 2\epsilon + \frac{1}{4} \\
 &\leq \frac{1}{2} \cdot \int \int_{\mathbb{R}^2} \log(|s - t|^2 + \epsilon) d\mu(s) d\mu(t) + (\alpha - 1) \log \epsilon + \log 16 + \frac{1}{4} + 2\epsilon \\
 &\leq \frac{1}{2} \cdot \int \int_{\mathbb{R}^2 - D} \log(|s - t|^2 + \epsilon) d\mu(s) d\mu(t) + \log \epsilon \cdot (\mu \times \mu)(D) + (\alpha - 1) \log \epsilon \\
 &\quad + \log 16 + \frac{1}{4} + 2\epsilon \\
 &\leq \frac{1}{2} \cdot \int \int_{\mathbb{R}^2 - D} \log(|s - t|^2 + \epsilon) d\mu(s) d\mu(t) + \log 16 + \frac{1}{4} + 2\epsilon
 \end{aligned}$$

Forcing  $\epsilon \rightarrow 0$  we arrive at the desired conclusion.  $\square$

#### 4. LOWER BOUND

**Lemma 4.1.**  $\lim_{k \rightarrow \infty} k^{-2} \cdot \log \left[ \Pi_{j=1}^k \frac{\Gamma(j+1)\Gamma(j)^2}{\Gamma(k+j)} \right] = -\log 4.$

*Proof.* Using Lemma 4.4 in [3],

$$\begin{aligned}
 \lim_{k \rightarrow \infty} k^{-2} \cdot \log \left[ \Pi_{j=1}^k \frac{\Gamma(j+1)\Gamma(j)^2}{\Gamma(k+j)} \right] &= \lim_{k \rightarrow \infty} k^{-2} \cdot \log \left[ \frac{\Pi_{j=1}^k \Gamma(j)^4}{\Pi_{j=1}^{2k} \Gamma(j)} \right] \\
 &= 4 \cdot \lim_{k \rightarrow \infty} (k^{-2} \cdot \log \Pi_{j=1}^k \Gamma(j) - 2^{-1} \log k) + \\
 &\quad 4 \cdot \lim_{k \rightarrow \infty} (-(2k)^{-2} \cdot \log \Pi_{j=1}^{2k} \Gamma(j) + 2^{-1} \log 2k) - \\
 &\quad 2 \log 2 \\
 &= -3 + 3 - 2 \log 2 \\
 &= -\log 4.
 \end{aligned}$$

$\square$

**Lemma 4.2.**  $\mathbb{H}^\alpha(x) \geq -\delta_0(x) \log 2 - \frac{1}{2} \log 288e + \frac{3}{4} + \int \int_{\mathbb{R}^2 - D} \log |s - t| d\mu(s) d\mu(t).$

*Proof.* Note that the inequality trivially holds when  $\mathbb{H}(x) = 1$ , i.e., when  $x$  has no eigenvalues. This follows from Proposition 4.4 in [3] and Lemma 3.7 of [1]. Also observe that the desired inequality is vacuously satisfied when the integral in question is  $-\infty$ . Thus, we assume without loss of generality, that  $x$  has a nontrivial point spectrum and  $\int \int_{\mathbb{R}^2 - D} \log |y - z| d\mu(y) d\mu(z) > -\infty$ .

Denote by  $G$  the group of diagonal unitaries and  $\mathbb{R}_{<}^k$  to be the set of all  $(t_1, \dots, t_k) \in \mathbb{R}^k$  such that  $t_1 < \dots < t_k$ . There exists a map  $\Phi : M_k^{sa}(\mathbb{C}) \rightarrow U_k/G \times \mathbb{R}_{<}^k$  defined almost everywhere on  $M_k^{sa}(\mathbb{C})$  such that for each  $x \in M_k^{sa}(\mathbb{C})$   $\Phi(x) = (h, z)$  where  $z$  is a diagonal matrix with real entries satisfying  $z_{11} < \dots < z_{kk}$  and  $h$  is the image of any unitary  $u$  in  $U_k/G$  satisfying  $uzu^* = x$ . By results of Mehta ([2]) the map  $\Phi$  induces a measure  $\mu$  on  $U_k/G \times \mathbb{R}_{<}^k$  given by  $\mu(E) = \text{vol}(\Phi^{-1}(E))$  and moreover,

$$\mu = \nu \times D_k \cdot \int_{\mathbb{R}_{<}^k} \Pi_{i < j} (t_i - t_j)^2 dt_1 \cdots dt_k$$

where  $D_k = \frac{\pi^{k(k-1)/2}}{\prod_{j=1}^k j!}$  and  $\nu$  is the normalized measure on  $U_k/G$  induced by Haar measure on  $U_k$ . Write  $\Theta_\epsilon(B_k)$  for the  $|\cdot|_2$   $\epsilon$ -neighborhood of the unitary orbit of  $B_k$  (as defined in subsection 2.2) and  $\Theta(B_k)$  for the unitary orbit of  $B_k$ . A matrix will be in  $\Theta_\epsilon(B_k)$  iff the sequence obtained by listing its eigenvalues in increasing order and according to multiplicity, differs from the similar sequence obtained from the eigenvalues of  $y_k$  by no more than  $\sqrt{k} \cdot \epsilon$  in  $\ell^2$  norm. In particular this will happen if the  $j$ th terms of the sequences differ by no more than  $\epsilon$ . constructed in Lemma 2.1. For sufficiently large  $k$ ,  $B_k \in \Gamma_R(x; m, k, \gamma)$ .

Suppose  $\epsilon_0 > \epsilon > 0$  and for each  $k$  define  $\Omega_k$  to be the intersection of

$$[b_{1k} - \epsilon, b_{1k} + \epsilon] \times \cdots \times [b_{kk} - \epsilon, b_{kk} + \epsilon]$$

with  $\mathbb{R}_{<}^k$ . Integrating according to the density given above it follows that  $\text{vol}(\Theta_\epsilon(B_k))$  exceeds

$$(3) \quad D_k \cdot \int_{\Omega_k} \prod_{i < j} (t_i - t_j)^2 dt_1 \cdots dt_k.$$

Recall the definitions of  $S_k$  and  $W_k$  in subsection 2.2 and denote by  $\Lambda_k$  the subset of  $\Omega_k$  consisting of all  $(t_1, \dots, t_k)$  satisfying  $|t_i - t_j| \geq |b_{ik} - b_{jk}|$ . (3) dominates

$$(4) \quad D_k \cdot \prod_{(i,j) \in W_k} (b_{ik} - b_{jk})^2 \int_{\Lambda_k} \prod_{(i,j) \in S_k} (t_i - t_j)^2 dt_1 \cdots dt_k.$$

Consider the map  $F : [-\epsilon, \epsilon]^k \cap \mathbb{R}_{<}^k \rightarrow \Lambda_k \subset \Omega_k$  given by  $F(t_1, \dots, t_k) = (t_1 + b_{1k}, \dots, t_k + b_{kk})$ . By a change of variables and Selberg's integral formula

$$\begin{aligned} \epsilon^{k^2} \cdot \prod_{j=1}^k \frac{\Gamma(j+1)\Gamma(j)^2}{\Gamma(k+j)} &< \int_{[-\epsilon, \epsilon]^k} \prod_{i < j} (t_i - t_j)^2 dt_1 \cdots dt_k \\ &= k! \cdot \int_{[-\epsilon, \epsilon]^k \cap \mathbb{R}_{<}^k} \prod_{i < j} (t_i - t_j)^2 dt_1 \cdots dt_k \\ &< k! \cdot (2\epsilon)^{k^2 - 2 \cdot \#S_k - k} \cdot \int_{[-\epsilon, \epsilon]^k \cup \mathbb{R}_{<}^k} \prod_{(i,j) \in S_k} (t_i - t_j)^2 dt_1 \cdots dt_k \\ &\leq k! \cdot (2\epsilon)^{k^2 - 2 \cdot \#S_k - k} \cdot \int_{\Lambda_k} \prod_{(i,j) \in S_k} (t_i - t_j)^2 dt_1 \cdots dt_k. \end{aligned}$$

From this it follows that  $\text{vol}(\Theta_\epsilon(B_k)) \geq (3) \geq (4) \geq C_k \cdot \epsilon^{\#S_k}$  where

$$C_k = D_k \cdot \prod_{(i,j) \in W_k} (b_{ik} - b_{jk})^2 \cdot (k!)^{-1} \cdot 2^{2 \cdot \#S_k + k - k^2} \cdot \prod_{j=1}^k \frac{\Gamma(j+1)\Gamma(j)^2}{\Gamma(k+j)}.$$

Thus for any  $\epsilon_0 > \epsilon > 0$  we have that  $P_\epsilon(\Theta(B_k)) \cdot L_k \cdot (3\epsilon)^{k^2} \geq \text{vol}(\Theta_\epsilon(B_k)) > C_k \cdot \epsilon^{2 \cdot \#S_k + k}$ . For large enough  $k$

$$2 \cdot \#S_k \leq 2 \left( \frac{([c_1 k] - \sqrt{k})^2}{2} + \sum_{j=2}^s \frac{[c_j k]^2}{2} \right) \leq \left( \sum_{j=1}^s [c_j k]^2 \right) - k \leq (1 - \alpha)k^2 - k$$

whence,  $2 \cdot \#S_k + k \leq (1 - \alpha)k^2$ . For any  $\epsilon_0 > \epsilon > 0$   $P_\epsilon(\Theta(B_k)) \geq C_k \cdot L_k^{-2} \cdot 3^{-k^2} \cdot \epsilon^{\alpha k^2}$ . Because  $\Theta(B_k)$  is locally isometric by Lemma 6.1 of [1] it follows that

$$H_{\epsilon_0}^{\alpha k^2}(\Gamma_R(x; m, k, \gamma)) \geq H_{\epsilon_0}^{\alpha k^2}(\Theta(B_k)) \geq C_k \cdot L_k^{-1} \cdot 3^{-k^2}.$$

$m$  and  $\gamma$  being arbitrary it follows from Lemma 4.4 of [3], Lemma 4.1, and (1) from section 2.2 that

$$\begin{aligned}
 \mathbb{H}^\alpha(x) &\geq \mathbb{H}_{\epsilon_0}^\alpha(x) = -\log 3 + \liminf_{k \rightarrow \infty} k^{-2} \cdot (\log C_k - \log L_k) \\
 &\geq -\log 3 - \frac{1}{2} \log 2\pi e + \liminf_{k \rightarrow \infty} \left[ k^{-2} \cdot \log C_k + \frac{1}{2} \cdot \log k \right] \\
 &\geq -\delta_0(x) \cdot \log 2 - \log 3 - \frac{1}{2} \log 2\pi e + \int \int_{\mathbb{R}^2-D} \log |s-t| d\mu(s) d\mu(t) + \\
 &\quad \liminf_{k \rightarrow \infty} \left[ k^{-2} \cdot \log D_k \cdot \prod_{j=1}^k \frac{\Gamma(j+1)\Gamma(j)^2}{\Gamma(k+j)} + \frac{1}{2} \cdot \log k \right] \\
 &\geq -\delta_0(x) \log 2 - \log 3 - \frac{1}{2} \log 2e + \frac{3}{4} + \int \int_{\mathbb{R}^2-D} \log |s-t| d\mu(s) d\mu(t) + \\
 &\quad \liminf_{k \rightarrow \infty} k^{-2} \cdot \log \left[ \prod_{j=1}^k \frac{\Gamma(j+1)\Gamma(j)^2}{\Gamma(k+j)} \right] \\
 &= -\delta_0(x) \log 2 - \frac{1}{2} \log 288e + \frac{3}{4} + \int \int_{\mathbb{R}^2-D} \log |s-t| d\mu(s) d\mu(t).
 \end{aligned}$$

□

**Remark 4.3.** Because  $\mathbb{H}^1(x) = \chi(x) + \frac{1}{2} \log \left( \frac{2}{\pi e} \right)$  it is clear that the lower bound of Lemma 2.1 is not sharp. This is also clear from the reduction to packing/covering number computations which in the microstate setting introduce non-sharp estimates.

**Example 4.4.** Suppose  $\mu = \sum_{j=1}^\infty \frac{1}{2^j} \cdot \delta_{\frac{1}{j}}$  where  $\delta_r$  is the Dirac mass concentrated at  $r$ . If  $x = x^* \in L^\infty \left( \langle \frac{1}{j} \rangle_{j=1}^\infty, \mu \right)$  is the identity multiplication operator then it follows from [1] and [4] that  $\mathbb{H}(x) = \delta_0(x) = 1 - \sum_{j=1}^\infty \frac{1}{4^j} = \frac{2}{3}$ . Moreover,  $-\infty < \sum_{i \neq j} \left( \frac{1}{2^{(i+j)}} \cdot \log \left| \frac{1}{i} - \frac{1}{j} \right| \right) < \infty$  so that by what preceded,  $-\infty < \mathbb{H}^{\frac{2}{3}}(x) < \infty$ .

## 5. FREE ADDITIVITY

In this section suppose  $x_1, \dots, x_n$  are selfadjoint elements of  $M$  and that for each  $1 \leq i \leq n$   $\mu_i$  is the Borel measure on  $sp(x_i)$  induced by  $\varphi$ . Set  $\alpha_i = \delta_0(x_i)$  and  $\beta = \alpha_1 + \dots + \alpha_n$ .

**Lemma 5.1.** If  $\{x_1, \dots, x_n\}$  is a freely independent family, then

$$K_1 \leq H^\beta(x_1, \dots, x_n) - \sum_{i=1}^n \int \int_{\mathbb{R}^2-D} \log |s-t| d\mu_i(s) d\mu_i(t) \leq K_2$$

where  $K_1 = -\frac{n}{2} \log 288e + \frac{3n}{4} - \beta \log 2$  and  $K_2 = n \log 16\sqrt{n} + \frac{n}{4}$ .

*Proof.* First for the lower bound on the difference. Suppose  $m \in \mathbb{N}$ ,  $1 > \epsilon_0, \gamma > 0$ , and  $R$  exceeds the maximum of the operator norms of any of the  $x_i$ . By Corollary 2.14 of [5] there exists an  $N \in \mathbb{N}$  such that if  $k \geq N$  and  $\sigma$  is a Radon probability measure on  $((M_k^{sa}(\mathbb{C}))_R)^n$  invariant under the  $(U_k)^{(n-1)}$ -action given by  $(\xi_1, \dots, \xi_n) \mapsto (\xi_1, u_1 \xi_2 u_1^*, \dots, u_{n-1} \xi_n u_{n-1}^*)$  where  $(u_1, \dots, u_{n-1}) \in (U_k)^{(n-1)}$ , then  $\sigma(\omega_k) > \frac{1}{2}$  where conclusion.

$$\omega_k = \{(\xi_1, \dots, \xi_n) \in ((M_k^{sa}(\mathbb{C}))_{R+1})^n : \{\xi_i\}_{i=1}^n \text{ are } \left(m, \frac{\gamma}{4^m}\right) \text{-free}\}.$$

The preceding section provided for each  $i$  a sequence  $\langle B_{ik} \rangle_{k=1}^\infty$  such that for any  $m' \in \mathbb{N}$  and  $\gamma' > 0$   $B_{ik} \in \Gamma_R(x_i; m', k, \gamma')$  for sufficiently large  $k$ . Also for any  $k$ ,  $\|B_{ik}\| \leq R$ . Write  $\Theta(B_{ik})$  for the

unitary orbit of  $B_{ik}$  and  $g_{ik}$  for the topological dimension of this orbit. The proof of Lemma 2.1 yielded constants  $C_{ik}$  for each  $1 \leq i \leq n$  such that for any  $\epsilon > 0$

$$P_\epsilon(\Theta(B_{ik})) \geq C_k \cdot L_k^{-1} \cdot 3^{-k^2} \epsilon^{\#S_{ik}-k^2}$$

where  $S_{ik} \leq (1 - \alpha_i)k^2$ . For each  $k \in \mathbb{N}$  denote by  $\mu_k$  the probability measure of  $((M_k^{sa}(\mathbb{C}))_{R+1})^n$  obtained by restricting  $\sum_{i=1}^n g_{ik}$ -Hausdorff measure (with respect to the  $|\cdot|_2$  norm) to the  $\sum_{i=1}^n g_{ik}$ -dimensional manifold  $T_k = \Theta(B_{1k}) \times \cdots \times \Theta(B_{nk})$  and normalizing appropriately.  $\mu_k$  is a Radon probability measure invariant under the  $(U_k)^{n-1}$ -action described above because such actions are isometric, whence  $\mu_k(\omega_k) > \frac{1}{2}$ . Set  $F_k = \omega_k \cap T_k$ . It is clear that  $\mu_k(F_k) = \mu_k(\omega_k) > \frac{1}{2}$  and for large enough  $k$ ,  $F_k \subset \Gamma_{R+1}(x_1, \dots, x_n; m, k, \gamma)$ .

$T_k$  is a locally isometric, smooth, compact manifold of dimension  $\sum_{i=1}^n g_{ik}$  (by locally isometric we means that for any  $\epsilon > 0$  any two open  $\epsilon$  balls of the metric space are isometric). Moreover by the preceding paragraph

$$P_\epsilon(T_k) \geq \Pi_{i=1}^n P_\epsilon(\Theta(B_{ik})) \geq \Pi_{i=1}^n C_{ik} \cdot L_k^{-1} \cdot 3^{-k^2} \epsilon^{\#S_{ik}-k^2}.$$

This estimate holds for all  $1 > \epsilon > 0$ . By Lemma 6.1 of [1]

$$H_{\epsilon_0}^{\beta k^2}(\Gamma_R(x_1, \dots, x_n; m, k, \gamma)) \geq H_{\epsilon_0}^{\beta k^2}(F_k) \geq H_{\epsilon_0}^{\sum_{i=1}^n (k^2 - S_{ik})}(F_k) \geq L_k^{-n} \cdot 3^{-nk^2} \cdot \frac{1}{2} \cdot \Pi_{i=1}^n C_{ik}.$$

Thus using the computations already made in Lemma 3.2,

$$\begin{aligned} \mathbb{H}_{\epsilon_0}^\beta(x_1, \dots, x_n; m, \gamma) &\geq \liminf_{k \rightarrow \infty} k^{-2} \log \left( L_k^{-n} \cdot 3^{-nk^2} \cdot \frac{1}{2} \cdot \Pi_{i=1}^n C_{ik} \right) \\ &\geq \sum_{i=1}^n \left[ -\log 3 + \liminf_{k \rightarrow \infty} k^{-2} \cdot \log(C_{ik} - \log L_k) \right] \\ &\geq -\frac{n}{2} \log 288e + \frac{3n}{4} - \beta \log 2 + \sum_{i=1}^n \int \int_{\mathbb{R}^2 - D} \log |s - t| d\mu_i(s) d\mu_i(t). \end{aligned}$$

Now for the upper bound. Suppose  $\epsilon, t > 0$ . Recalling the proof of Lemma 3.1 we can produce  $m \in \mathbb{N}$  and  $\gamma > 0$  such that

$$\begin{aligned} H_\epsilon^{\beta k^2}(\Gamma(x_1, \dots, x_n; m, k, \gamma)) &\leq H_\epsilon^{\beta k^2}(\Pi_{i=1}^n \Theta_t(A_{ik})) \\ &\leq K_{\frac{\epsilon}{2}}(\Pi_{i=1}^n \Theta_t(A_{ik})) \cdot \epsilon^{\beta k^2} \\ &\leq \left[ \Pi_{i=1}^n K_{\frac{\epsilon}{2\sqrt{n}}}(\Theta_t(A_{ik})) \right] \cdot \epsilon^{\beta k^2} \\ &\leq \left[ \Pi_{i=1}^n P_{\frac{\epsilon}{4\sqrt{n}}}(\Theta_t(A_{ik})) \right] \cdot \epsilon^{\beta k^2} \\ &\leq \left[ \Pi_{i=1}^n \text{vol}(\Theta_{t+\frac{\epsilon}{4\sqrt{n}}}(A_{ik})) \right] \cdot \frac{(4\sqrt{n})^{nk^2} \epsilon^{(\beta-n)k^2}}{L_k^n}. \end{aligned}$$

Thus,

$$\mathbb{H}_\epsilon^\beta(x_1, \dots, x_n; m, \gamma) \leq \sum_{i=1}^n \limsup_{k \rightarrow \infty} k^{-2} \cdot [\log(\text{vol} \Theta_{t+\frac{\epsilon}{4\sqrt{n}}}(A_{ik})) - \log L_k] + \log(4\sqrt{n}) + (\alpha_i - 1) \log \epsilon.$$



This bound being independent of  $m$  and  $\gamma$ , letting  $\epsilon \rightarrow 0$ , and using the computation already made in Lemma 3.1, we conclude that

$$\mathbb{H}^\beta(x_1, \dots, x_n) \leq \left( \sum_{i=1}^n \int \int_{\mathbb{R}^2-D} \log |s - t| d\mu_i(s) d\mu_i(t) \right) + n \log 16\sqrt{n} + \frac{n}{4}.$$

□

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